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Single and multiple consecutive permutation motif search

Djamal Belazzougui*, Adeline Pierrot**,
Mathieu Raffinot**, and Stéphane Vialette***

Abstract: Let t be a permutation (that shall play the role of the *text*) on $[n]$ and a pattern p be a sequence of m distinct integer(s) of $[n]$, $m \leq n$. The pattern p occurs in t in position i if and only if $p_1 \dots p_m$ is order-isomorphic to $t_i \dots t_{i+m-1}$, that is, for all $1 \leq k < \ell \leq m$, $p_k > p_\ell$ if and only if $t_{i+k-1} > t_{i+\ell-1}$. Searching for a pattern p in a text t consists in identifying all occurrences of p in t . We first present a forward automaton which allows us to search for p in t in $O(m^2 \log \log m + n)$ time. We then introduce a Morris-Pratt automaton representation of the forward automaton which allows us to reduce this complexity to $O(m \log \log m + n)$ at the price of an additional amortized constant term by integer of the text. Both automata occupy $O(m)$ space. We then extend the problem to search for a set of patterns and exhibit a specific Aho-Corasick like algorithm. Next we present a sub-linear average case search algorithm running in $O\left(\frac{m \log m}{\log \log m} + \frac{n \log m}{m \log \log m}\right)$ time, that we eventually prove to be optimal on average.

1 Introduction

Two sequences are *order-isomorphic* if the permutations required to sort them are the same. A sequence p is said to be a *pattern* (or *occurs*) within a sequence t if t has a *subsequence* that is order-isomorphic to p . Pattern involvement *permutations* and sequences has now become a very active area of research [10]. However, only few results on the complexity of finding patterns in permutations and sequences are known. It appears to be a difficult problem to decide of two given permutations π and σ whether σ occurs in π , and in this generality the problem is NP-complete [7]. For $\sigma \in S_m$ and $\pi \in S_n$, the $O(n^m)$ time brute-force algorithm was improved to $O(n^{0.47m+o(m)})$ time in [2]. There are several ways in which this notion of permutation patterns may be generalized, and we focus here on *consecutive patterns* (*i.e.* the match is required to consist of contiguous elements) [10]. A sequence p is said to be a *consecutive pattern* or *consecutively occurs* within a sequence t if t has a *substring* that is order-isomorphic to p . Searching for a pattern p in a text t consists in identifying all occurrences of p in t . Recently, using a modification of the classical Knuth-Morris-Pratt string matching algorithm, a $O(n + m \log m)$ time algorithm has been proposed for checking if a given sequence t of length n contains a substring which is order-isomorphic to a given pattern p of length m [11]. The time complexity reduces to $O(n + m)$ time under the assumption that the symbols of the pattern can be sorted in $O(m)$ time.

The set of all integers from 1 to n is written $[n]$. Let t be a permutation of length n and p be a sequence of $m \leq n$ distinct integers in $[n]$. First we present

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a forward automaton which allows us to search for p in t in $O(m^2 \log \log m + n)$ time. Next, we introduce a Morris-Pratt automaton representation [9] of the forward automaton which allows us to reduce this complexity to $O(m \log \log m + n)$ at the price of an additional amortized constant term by integer of the text. Both automata occupy $O(m)$ space. We then extend the problem to search for a set of patterns and exhibit a specific Aho-Corasick like algorithm. Finally we present a sub-linear average case search algorithm running in $O(n \log m / \log \log m)$ time that we eventually prove to be optimal on average. Both lower and upper bounds assume all text permutations to be equiprobable and all integer values in a pattern to be distinct.

Let us define some notations. The set of all permutations on $[n]$ is denoted by S_n . Let $\Sigma_n = [n]$. Abusing notations, we consider in this paper permutations of S_n as strings without symbol repetition, and we denote by Σ_n^* the set of all strings without symbol repetition (including the empty string), where each symbol is an integer in $[n]$. A *prefix* (resp. *suffix*, *factor*) u of p is a string such that $p = uw, w \in \Sigma_n^*$. (resp. $p = wu, w \in \Sigma_n^*$, $p = wuz, w, z \in \Sigma_n^*$). We also denote $|w|$ the number of integer(s) in a string $w, w \in \Sigma_n^*$. We eventually denote p^r the reverse of p , that is, the string formed by the symbols of p read in the reverse order. We denote by p^\equiv the set of words of Σ_n^* which are order-isomorphic to p .

The following property is useful for designing automaton transitions.

Property 1. Let $p = p_1 \dots p_m \in \Sigma_n^*$ and $w = w_1 \dots w_\ell \in \Sigma_n^*$, $\ell < m$, such that w is order-isomorphic to $p_1 \dots p_\ell$, and let $\alpha \in \Sigma$. Testing if $w\alpha$ is order-isomorphic to $p_1 \dots p_\ell p_{\ell+1}$ can be performed in constant time storing only a pair of integers.

Proof. The pair of integers (x_1, x_2) is determined as follows: $x_1 \leq \ell$ is the position of the largest number p_{x_1} in $p_1 \dots p_\ell$ which is smaller than $p_{\ell+1}$, if any. Otherwise, we fix x_1 arbitrarily to $-\infty$. Let $x_2 \leq \ell$ be the position of the smallest integer p_{x_2} in $p_1 \dots p_\ell$ which is larger than $p_{\ell+1}$, if any. Otherwise, we fix x_2 to $+\infty$. Now, it suffices to test if $w_{x_1} < \alpha < w_{x_2}$ to verify if $w\alpha$ is order-isomorphic to $p_1 \dots p_{\ell+1}$ \square

We define a function $\text{rep}(p = p_1 \dots p_m, j)$ which returns a pair of integers (x_1, x_2) that represents the pair defined in property 1 for the prefix of length j of a motif p .

2 Tools

Before proceeding, we first describe some useful data structures we shall use as basic subroutines of our algorithms. The problem called *predecessor search problem* is defined as follows: given a set $S = \{x_1, x_2, \dots, x_n\} \subset [u]$ (u is called the size of the universe), we support the following query: given an integer y return its predecessor in the set S , namely the only element x_i such that $x_i \leq y \leq x_{i+1}$ ¹. In addition, in the dynamic case, we also support updates: add or remove an

¹ By convention, if all the elements of S are smaller than y , then return $-\infty$ and if they are larger than y then return x_n

element from the set S . The standard data structures to solve the predecessor search are the balanced binary search trees [1,5]. They use linear space and support queries and updates in worst-case $O(\log n)$ time. However, there exists better data structures that take advantage of the structure of the integers to get better query and update time. Specifically, the Van-Emde-Boas tree [14] supports queries and updates in (worst-case) time $O(\log \log u)$ using $O(u)$ space. Using randomization, the y-fast trie achieves linear space with queries supported in time $O(\log \log u)$ and updates supported in randomized $O(\log \log u)$ time. The problem has received series of improvements which culminated with Andersson and Thorup's result [4]. They achieve linear space with queries and updates supported in $O(\min(\log \log u, \sqrt{\frac{\log n}{\log \log n}}))$ (the update time is still randomized).

A special case occurs when space n is available and the set of keys S is known to be smaller than $\log^c n$ for some constant c . In this case all operations are supported in worst-case constant time using the atomic-heap [15].

3 Forward search automaton

The problem we consider is to search for a motif p in a permutation t without preprocessing the text itself. By analogy to the simpler case of the direct search of a word p in text t , we build an automaton that recognizes $(\Sigma_n^*)p^\equiv$. We then prove its size to be linear in the length of the pattern.

We formally define our forward search automaton $\mathcal{FD}(p)$ built on $p = p_1 \dots p_m$ as follows: (i) $m + 1$ states corresponding to each prefix (including the empty prefix) of p , state 0 is initial, state m is terminal; (ii) m forward transitions from state j to $j + 1$ labeled by $\text{rep}(p, j + 1)$; (iii) bt backward transitions $\delta(x, [i, j])$, where x numbers a state, $0 \leq x \leq m$, $i \in 1, \dots, x \cup -\infty$, $j \in 1, \dots, x \cup +\infty$, defined the following way: $\delta(x, [i, j]) = q$ if and only if for all $p_i < \alpha < p_j$ (resp. $k = \alpha < p_j$ if $i = -\infty$, $p_i < \alpha$ if $j = +\infty$), the longest prefix of p that is order-isomorphic to a suffix of $p_1 \dots p_x \alpha$ is $p_1 \dots p_q$. We also impose some constraints on outgoing transitions. Let x be a given state corresponding to the prefix $p_1 \dots p_x$.

Let us sort all $p_i, 1 \leq i \leq x$ and consider the resulting order $p_{i_0} = -\infty < p_{i_1} < \dots < p_{i_k} < +\infty = p_{i_{k+1}}$. We build one outgoing transition for each interval $[p_{i_j}, p_{i_{j+1}}]$, excepted if $p_{i_{j+1}} = p_{i_j} + 1$. Also we merge transitions from the same state to the same state that are labeled by consecutive intervals.

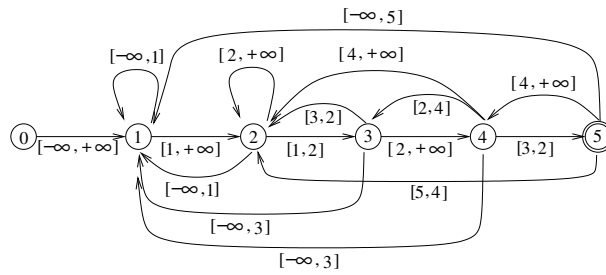


Fig. 1. Forward automaton built on $p = 4\ 12\ 6\ 16\ 10$. State 0 is initial and state 5 is terminal.

It is obvious that the resulting automaton recognizes a given pattern in a permutation by reading one by one each integer and choose the appropriate transition. Figure 1 shows such an automaton. The main result on the structure of the forward automaton is the following.

Lemma 1. *The number of transitions of the forward automaton built on $p_1 \dots p_m$ is linear in m .*

Lemma 1 combined with the fact that the outgoing transitions from each state q are sorted accordingly to the closest proximity to q of their arrival state leads to the following lemma.

Lemma 2. *Searching for a consecutive motif $p = p_1 \dots p_m$ in a permutation $t = t_1 \dots t_n$ using a forward automaton built on p takes $O(n)$ time.*

We can build the forward automation in $O(m^2 \log \log m)$ time. However, we defer the proof of this construction for the following reason. This $O(m^2 \log \log m)$ complexity might be too large for long patterns. Nevertheless, we show below that we can compute in a first step a type of Morris-Pratt coding of this automaton which can either (a) be directly used for the search for the pattern in the text and will preserve the linear time complexity at the cost of an amortized constant term by text symbol, or (b) be developed to build the whole forward automaton structure.

Therefore we present and build a new automaton \mathcal{MP} that is a Morris-Pratt representation of the forward automaton. The idea is to avoid building all backward transitions by only considering a special backward single transition from each state $x, x > 0$ named *failure* transition. We formally define our automaton $\mathcal{MP}(p)$ built on $p = p_1 \dots p_m$ the following way: (i) $m + 1$ states corresponding to each prefix (including the empty prefix) of p , state 0 is initial, state m is terminal; (ii) m forward transitions from state j to $j + 1$ labeled by $\text{rep}(p, j + 1)$; (iii) m failure transitions (non labeled) defined by: a failure transition connects a state $j > 0$ to a state $k < j$ if and only if $p_1 \dots p_k$ is the largest order-isomorphic border of $p_1 \dots p_j$. Figure 2 shows such an \mathcal{MP} automaton.

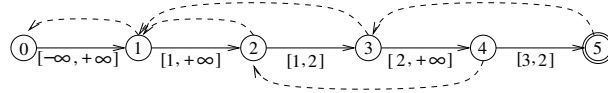


Fig. 2. \mathcal{MP} automaton built on $p = (4, 12, 6, 16, 10)$. State 0 is initial and state 5 is terminal. Backward transitions are failure transitions.

Reading a text t through the MP representation of the forward automaton is performed the following way. Let us assume we reached state $x < m$ and we read a symbol t_i at position i of the text. Let $[k, \ell] = \text{rep}(p, x + 1)$. If $t_i \in [t_{i-m+k}, t_{i-m+\ell}]$ we follow the forward transition and the new current state is $x + 1$. Otherwise, we *fail* reading t_i from x and we retry from state $q = \text{fail}(x)$ and so-on until (a) either q is undefined, in which case we start again from state 0, either (b) a forward transition from q to $q + 1$ works, in which case the next current state is $q + 1$.

Lemma 3. *Searching for a pattern p in a text $t_1 \dots t_m$ using the Morris-Pratt representation of the forward automaton built on p is $O(n)$ time.*

In order to prove lemma 3 we need to focus on the classical notion of border that we extend to our framework.

Definition 1. *Let $p \in \Sigma_n^*$. A border of p is a word $w \in \Sigma_n^*$, $|w| < |p|$ that is order-isomorphic to a suffix of p but also order-isomorphic to a prefix of p .*

The construction of the forward automaton relies on the maximal border of each prefix that is followed by an appropriate integer in the pattern. The Morris-Pratt approach is based on the following property:

Property 2. A border of a border is a border.

This property allows us to replace the direct transition of the forward algorithm by a search along the borders, from the longest to the smallest, to identify the longest one that is followed by the appropriate integer. We prove now that we can build the Morris-Pratt representation of the forward automaton efficiently.

Lemma 4. *Building an Morris-Pratt representation of the forward automaton on a consecutive motif $p = p_1 \dots p_m$ can be performed in (worst-case) $O(m \log \log m)$ time.*

Lemma 3 and 4 allow us to state the main theorem of this section.

Theorem 1. *Searching for a consecutive motif $p = p_1 \dots p_m$ in a permutation $t = t_1 \dots t_n$ can be done in $O(m \log \log m + n)$ time.*

The Morris-Pratt representation of the forward automaton permits to search directly in the text at the price of larger amortized complexity (considering the constant hidden by the O notation) than that required by searching with the forward automaton directly. If the real time cost of the search phase is an issue, the forward automaton can be built from its Morris-Pratt representation as follows.

Property 3. Building the forward automaton of a consecutive motif $p = p_1 \dots p_m$ can be performed in $O(m^2 \log \log m)$ time.

An interesting point is that the construction of the forward automaton from its Morris-Pratt representation can also be performed in a lazy way, that is, when reading the text. The missing transitions are then built *on the fly* when needed.

4 Multiple worst case linear motif searching

We can extend the previous problem defined for a single pattern to a set of patterns S . We note by d the number of patterns, by m the total length of the patterns and by r the length of the longest pattern. For this problem we adapt the Aho-Corasick automaton [3] (or \mathcal{AC} automaton for short). The \mathcal{AC} automaton

is a generalization of the \mathcal{MP} automaton to a set of multiple patterns. We note by P the set of prefixes of strings in S . In order to simplify the description we will assume that the set of patterns S is prefix-free. That is, we will assume that no pattern is prefix of another. Extending the algorithm to the case where S is non-prefix free, should not pose any particular issue. The states of the \mathcal{AC} automaton are defined in the same way as in the \mathcal{MP} automaton. Each state t in the \mathcal{AC} automaton corresponds uniquely to a string $p \in P$. The forward transitions are defined as follows: there exists a forward transition connecting state s to each state corresponding to an element $pc \in P$ (where c is a single symbol). Thus this definition of the forward transitions matches essentially the definition of the forward transitions in the \mathcal{MP} automaton. The failure transitions are defined as follows: a failure transition a state s corresponding a string p to the state s' corresponding to the longest string q such that $q \in P$ and $q \neq p$. The matching using the \mathcal{AC} automaton is done in the same way as in the \mathcal{MP} automaton using the forward and failure transitions.

Our extension of the \mathcal{AC} automaton. We could use exactly the same algorithm as the one used previously for our variant of the \mathcal{MP} automaton with few differences. We describe our modification to \mathcal{AC} automaton to adapt it to the case of consecutive permutation matching. An important observation is that we could have two or more elements of P that are both of the same length and order-isomorphic. Those two elements should have a single corresponding state in the \mathcal{AC} automaton. Thus, if two or more elements of P are order-isomorphic then we keep only one of them. For the forward transitions, we can associate a pair of positions (x_1, x_2) to each forward transition. Then we can check which transition is the right one by checking the condition $t_{i-m+x_1} < t_i < t_{i-m+x_2}$ for every pair (x_1, x_2) and take the corresponding transition. The main problem with this approach is that the time taken would grow to $O(d)$ time to determine which transition to take which can lead to a large complexity if d is very large. Our approach will instead be based on using a binary search tree (or more sophisticated predecessor data structure). With the use of a binary search tree, we can achieve $O(\log r)$ time to decide which transition to take. More precisely, each time we read t_i we insert the pair (t_i, i) into the binary search tree. The insertion uses the number t_i as a key. Now suppose that we only pass through forward transitions. Then a transition at step i is uniquely determined by: (1) the current state s corresponding to an element $p \in P$; (2) the position of the predecessor of t_i among $t_{i-|p|} \dots t_{i-1}$.

To determine the predecessor of t_i among $t_{i-|p|} \dots t_{i-1}$, the binary search tree should contain precisely the $|p|$ pairs corresponding to $t_{i-|p|} \dots t_{i-1}$. If the predecessor of t_i in the binary search tree is a pair (t_j, j) , we then conclude that the element $p[|p| - j + 1]$ is the predecessor of t_i in p .

In order to maintain the binary search tree we must do the following actions during passing through a failure or a forward transition: (1) whenever we pass through a forward transition at a step i we insert the pair (t_i, i) ; (2) whenever we pass through a failure transition from a state corresponding to a prefix p_1

to a state corresponding to a prefix p_2 , then we should remove from the binary tree all the pairs corresponding to the symbols $t_{i-|p_1|} \dots t_{i-|p_2|}$.

It should be noted that each removal or insertion of a pair into the binary search tree takes $O(\log r)$ time. The upper bound $O(\log r)$ comes from the fact that we never insert more than r elements in the binary search tree. Since in overall we are doing $O(n)$ insertions or removals, the amortized time should simplify to $O(n \log r)$. Finally if we replace binary search tree with a more efficient predecessor data structure, we will be able to achieve randomized time $O(n \cdot t)$ where $t = \min(\log \log n, \sqrt{\frac{\log r}{\log \log r}}, d)$ is the time needed to do an operation on the predecessor data structure (see section 2 for details). We use the linear space version of the predecessor data structure which guarantees only randomized performance but uses $O(r)$ additional space only. We thus have the following theorem :

Theorem 2. *Searching for set of d consecutive motifs of maximal length r and whose \mathcal{AC} automaton has been built and where the longest pattern is of length r can be done in randomized $O(nt)$ time, where $t = \min(\log \log n, \sqrt{\frac{\log r}{\log \log r}}, d)$.*

Preprocessing. We now show that the preprocessing phase can be done in worst-case $O(m \log \log r)$ time. As before our starting point will be to sort all the patterns and reduce the range of symbols of each pattern of length ℓ from range $[n]$ to the range $[1..\ell]$. This takes worst-case time $O(m \log \log r)$.

Recall that two or more elements of P of the same length and order-isomorphic should be associated with the same state in the \mathcal{AC} automaton. In order to identify the order-isomorphic elements of P , we will carry a first step called normalization. It consists in normalizing each pattern. A pattern p is normalized by replacing each symbol p_j by the pair $\text{rep}(p = p_1 \dots p_{j-1}, j)$ (consisting in the positions of the predecessor and successor among symbols $p_1 \dots p_{j-1}$). This can be done for all patterns in total $O(m \log \log r)$ time. In the next step, we build a trie on the set of normalized patterns. This takes linear time. The trie naturally determines the forward transitions. More precisely any node in the trie will represent a state of the automaton and the labeled trie transitions will represent follow transitions.

Note that unlike the forward automaton (or the \mathcal{MP} automaton) there could be more than one outgoing forward transition from each node. In order to encode the outgoing transition from each node, we will make use of a hash table that stores all the transitions outgoing from that node. More precisely for each transition labeled by the pair $\text{rep}(p = p_1 \dots p_{j-1}, j)$ and directed to a state q , the hash table will associate the key p_1 associated with the value q . Now that the next transitions have been successfully built, the final step will be to build the failure transitions and this takes more effort. The construction of the failure transitions can also be done in worst-case $O(m \log \log r)$ time, but for lack of space we defer the details to the appendix.

We thus have the following theorem:

Theorem 3. *Building the AC automaton for a set of d consecutive motifs of total length m and where the longest motif is of length r can be done in worst-case $O(m \log \log r)$ time.*

5 Single sublinear average-case motif searching

Algorithm forward takes $O(n + m \log \log m)$ time in the worst case time but also on average. We present now a very simple and efficient average case-algorithm which takes $O(\frac{m \log m}{\log \log m} + n \frac{\log m}{m \log \log m})$ time.

In order to search for a pattern p in t , we first build a tree T of all isomorphic-order factors of p^r of length $\frac{3.5 \log m}{\log \log m}$. T is built by inserting each factor one after the other in a tree and building the corresponding path if it does not already exist. The construction of this tree requires $O(\frac{m \log m}{\log \log m})$ time (details are given below). The search phase is performed through a window of size m that is shifted along the text. For each position of this window, $b = \frac{3.5 \log m}{\log \log m}$ symbols are read backward from the end of the window in the tree T . Two cases may occurs: (i) either the factor is not recognized as a factor of p^r . This means that no occurrence of p might overlap this factor and we can surely shift the search window after the last symbol of this factor; (ii) either the factor is recognized, in which case we simply check if the motif is present using a naive $O(m)$ algorithm, and we repeat this test for the next $O(m/2)$ symbols. This might require $O(m^2/2)$ steps in the worst case.

Let us analyze the average complexity of our algorithm, in the following model: all text permutations are considered to be equiprobable, all integer values in a pattern are distinct.

We count the average number of symbol comparisons required to shift the search window of $m/2$ symbols to the right. As there are $2n/m$ such segments of length $m/2$ symbols in n , we will simply multiply the resulting complexity by $2n/m$ to gain the whole average complexity of our algorithm.

There might be $O(b!)$ distinct motifs that could appear in the text while this number is bounded by $m - b + 1$ in the pattern (one by position). Thus, with a probability bounded by $\frac{m-b+1}{b!}$ we will recognize the segment of the text as a factor of p and enter case 2. In which case, moving the search window of $m/2$ symbols to the right using the naive algorithm will require $O(m^2/2)$ worst case time.

In the other case which occurs with probability at least $1 - \frac{m-b+1}{b!}$, shifting the search window by $m/2$ symbols to the right only requires reading b numbers.

The average complexity (in terms of number of symbol reading and comparisons) for shifting by $m/2$ symbols is thus (upper) bounded by $A = O((m^2/2)^{\frac{m-b+1}{b!}} + b(1 - \frac{m-b+1}{b!}))$ and the whole complexity by $O((2n/m)A)$. By expanding and simplifying A we get that $A = O(b + O(m^3/2b!))$. Now using the famous Stirling approximation $\ln(m!) = m \ln m - m + O(\ln m)$, it is not difficult to prove that $b! = 2^{b \log b - b \log e + O(\log b)} = \Omega(m^3)$ and thus $A = O(b)$ and the whole average time complexity (in terms of number of symbol reading and comparisons) turns out to be $O(\frac{n \log m}{m \log \log m})$.

Implementation details. The tree T can actually be built in $O(\frac{m \log m}{\log \log m})$ time by using appropriate data structures. Recall that the tree T recognizes all the factors of p^r of length $\frac{3.5 \log m}{\log \log m}$. To implement T , we use the same \mathcal{AC} automaton presented in previous section to build the tree T , but with two differences: we only need forward transitions and the length of any pattern is bounded by $\frac{\log m}{\log \log m}$. Thus the cost is upper bounded by $O(\frac{m \log m}{\log \log m} \cdot t)$, where t is the time needed to do an operation on the predecessor data structure (maximum of the times needed for inserts/deletes and searches). We now turn our attention to the cost of the matching phase. From the previous section, we know that the total complexity in terms of number of symbol reading and comparisons is $O(\frac{n \log m}{m \log \log m})$. The total cost of the matching phase is dominated by the multiplication of the total number of text symbols read multiplied by the cost of a transition in the \mathcal{AC} automaton which itself is dominated by the time to do an operation on a predecessor data structure. The total cost of the matching phase is thus $O(\frac{n \log m}{m \log \log m} \cdot t)$, where t is the time needed to do an operation on the predecessor data structure.

Now the performance of both matching and building phases crucially depend on the used predecessor data structure. If a binary search tree is used then $t = O(\log \frac{\log m}{\log \log m}) = O(\log \log m)$ and the total matching time becomes $O(nt) = O(n \log \log m)$, and the total building time becomes $O(m \log m)$. However, we can do better if we work in the word-RAM model. Namely, we can use the atomic-heap (see section 2) which would add additional $o(m)$ words of space and support all operations (queries, inserts and deletes) in constant time on sets of size $\log^{O(1)} m$. In our case, we have a set of size $O(\frac{\log m}{\log \log m})$ and thus the operations can be supported in constant time. We thus have the following theorem:

Theorem 4. *Searching for a consecutive motif $p = p_1 \dots p_m$ in a permutation $t = t_1 \dots t_n$ can be done in average $O(\frac{m \log m}{\log \log m} + \frac{n \log m}{m \log \log m})$ time.*

6 Average optimality

We prove in this section a lower bound on the average complexity of any consecutive motif matching algorithm. The proof of this bound is inspired by that of Yao [16] which proved an average lower bound for matching a pattern of length m in a text of length n . We prove in our case of interest an average lower bound of $\Omega(\frac{n \log m}{m \log \log m})$ considering all permutations over $[n]$ to be equiprobable. As this average complexity is reached by the algorithm we designed in the previous section, this bound is tight.

We begin to circumscribe our problem on small segments of length $2m - 1$ of the text into which we search for. Precisely, following [16, 12], we divide our text in $\lfloor n/(2m - 1) \rfloor$ contiguous and no-overlapping segments $s_i, 1 \leq \lfloor n/(2m - 1) \rfloor$, such that $s_i(t) = t_{(2m-1)(i-1)+1} \dots t_{(2m-1)i}$. When searching for a pattern in t , there might be occurrences overlapping two blocks. But as we are interested on a lower bound, the following lemma allows us to focus on all segments.

Lemma 5. *A lower bound for finding a pattern p inside all segments $s_i(t)$ is also a lower bound to the problem of searching for all occurrences of p in t .*

We now prove that instead of focusing on all segments $s_i(t)$, we can focus on obtaining a lower bound to search p in any single segment and then extend the lower bound on searching for p inside this segment to searching for p inside all segments, and thus, using the previous lemma, to the whole text.

Lemma 6. *The average time for searching for p inside all segments $s_i(t)$ is $\lfloor n/(2m-1) \rfloor$ times the average time for searching for p inside any such segment.*

Let $E(t)$ be the average complexity for searching p in any segments. Using the previous lemma, the whole average complexity is $\sum_{i=1}^{\lfloor n/(2m-1) \rfloor} E(t) = \lfloor n/(2m-1) \rfloor E(t) = \Omega(n/m)E(t)$.

We now prove a lower bound for $E(t)$, which, using the two previous lemma, gives us a lower bound for the whole problem. Let $\mathcal{P}_m(\ell)$ the number of permutations of size m that can be discarded using a sliding window of size m over a text of size $2m-1$ and checking only $0 < \ell \leq m$ positions in this window.

Lemma 7 (Counting lemma). *Let $0 < \ell \leq m$. Then $|\mathcal{P}_m(\ell)| \leq m! \left(1 - \frac{1}{\ell!}\right)^{\lceil \frac{m-1}{\ell^2} \rceil}$.*

Let us consider now the whole set S_m of permutations of length m which contains $m!$ such permutations. Given $1 < l(m) \leq m$, this set is the union of two distinct set $\mathcal{P}_m(\ell)$ and $S_m \setminus \mathcal{P}_m(\ell)$, that is the set of motifs discarded by a certificate of length l (or by l accesses) and the others. For all pattern in $\mathcal{P}_m(\ell)$, the average complexity to be discarded is counted 1. For any other motif in $S_m \setminus \mathcal{P}_m(\ell)$, the average complexity is at least $l+1$.

The average complexity for discarding all patterns in S_m is thus $C(m) = \frac{|\mathcal{P}_m(\ell)| + (m! - |\mathcal{P}_m(\ell)|)(l+1)}{m!}$. We aim to find $l(m)$ that maximizes this expression when m grows, which will provide us a lower bound for the whole average complexity. Now let us consider a fixed $l(m)$. We need to lower bound $C(m)$. As $C(m)$ decreases when $\mathcal{P}_m(\ell)$ increases, this lower bound is minimal when $\mathcal{P}_m(\ell)$ is as large as possible. Then, as the counting lemma states that $|\mathcal{P}_m(\ell)| \leq m! \left(1 - \frac{1}{\ell!}\right)^{\lceil \frac{m-1}{\ell^2} \rceil}$, $C(m)$ is minimal when $|\mathcal{P}_m(\ell)| = m! \left(1 - \frac{1}{\ell!}\right)^{\lceil \frac{m-1}{\ell^2} \rceil}$. We now arbitrarily impose $98/100 \leq \frac{|\mathcal{P}_m(\ell)|}{m!} \leq 99/100$. With the left constraint, $C(m) \geq l - 98/100l + 1 = \Omega(l)$. We want to compute $l(m)$ such that $98/100 \leq \frac{|\mathcal{P}_m(\ell)|}{m!} = \left(1 - \frac{1}{\ell!}\right)^{\lceil \frac{m-1}{\ell^2} \rceil} \leq 99/100$. Let us impose $\lceil \frac{m-1}{\ell^2} \rceil \times \frac{1}{\ell!} \leq 1/10$ (ineq.1). This allows us to approximate our equation using the classical formula $(1+x)^a = 1 + ax + \frac{a(a-1)}{2!}x^2 + \dots + \frac{a!}{n!(a-n)!}x^n = 1 + ax + \gamma$ where $a = \lceil \frac{m-1}{\ell^2} \rceil$, $x = \frac{-1}{\ell!}$ and $\gamma = \sum_{i=2}^n \frac{a!}{i!(a-i)!}x^i$. It is easy to see that inequality (1) implies that γ converges and is dominated by its first term which is bounded $\frac{a(a-1)}{2!}x^2 \leq 1/200$. We thus deduce that $(1+x)^a \in [1+ax, 1+ax+1/200]$ which implies that $(1+x)^a - 1/200 \leq 1+ax \leq (1+x)^a$. From $(1+x)^a = \frac{|\mathcal{P}_m(\ell)|}{m!} \in [\frac{98}{100}, \frac{99}{100}]$, we obtain $\frac{98}{100} - \frac{1}{200} \leq 1+ax \leq \frac{99}{100}$. By replacing a and x in $1+ax$ we get $\frac{98}{100} - \frac{1}{200} = 195/200 \leq 1 - \lceil \frac{m-1}{\ell^2} \rceil \times \frac{1}{\ell!} \leq 99/100$. We prove in appendix

that $l = \frac{b \log m}{\log \log m}$ with $b = 1 + o(1)$ verify these two inequalities and inequality (1). Thus $\Omega(\frac{n \log m}{m \log \log m})$ is a lower bound of the whole average complexity for searching for a consecutive motif in a permutation.

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Appendix

Proof (Of Lemma 1).

Point 1. We adapt the technique of [13] to our framework. Let $q = \delta(x, [i, j])$ a backward transition from x to q such that $q \geq 2$. Then $p_1 \dots p_{q-1}$ is order-isomorphic to the suffix of $p_1 \dots p_x$ of length $q - 1$. But either (a) $p_1 \dots p_q$ is not order-isomorphic with $p_1 \dots p_x$, or (b) $x = m$ (x is the last state of the automaton. Let $\ell = x - q$. We prove now *a contrario* that no other backward transition $q' = \delta(x', [i', j'])$ such that $q' \geq 2$ can accept the same difference $\ell' = x' - q' = \ell$. Let $q' = \delta(x', [i', j'])$ be such a transition and consider without lost of generality that $2 \leq q' < q$. Then $p_1 \dots p_{q'-1}$ would be order-isomorphic to the suffix of $p_1 \dots p_{x'}$ of length $q' - 1$, and $p_1 \dots p_{q'}$ must not be order-isomorphic to $p_1 \dots p_{x'} p_{x'+1}$. However, as $2 \leq q' < q$, $p_1 \dots p_{q'}$ is a prefix of $p_1 \dots p_{q-1}$, and as $\ell' = \ell$, $p_1 \dots p_{q-1}$ is order-isomorphic to the prefix of $p_1 \dots p_x$ of length q' , which is exactly $p_1 \dots p_{x'} p_{x'+1}$. This leads to a contradiction and for a given $1 \leq \ell < m$, there exists at most one backward transition $q = \delta(x, [i, j])$, $q \geq 2$ such that $x - q = \ell$. This bounds the number of such backward transition to $m - 2$. Let $N(x)$ be the number of backward transitions $q = \delta(x, [i, j])$ from x such that $q \geq 2$.

Point 2. We consider now all backward transitions $1 = \delta(x, [i, j])$ reaching state 1. We denote such a transition a 1-transition. Note that state 0 is never reached by any transition because any two integers are always order-isomorphic. The key observation is that from each state x source of the transition, the number of such 1-transitions from x is bounded by $N(x) + 2$. This is true since 1-transitions and other transitions must be interleaved to cover $[-\infty, +\infty]$. Therefore, as the total number of $N(x)$ is bounded by $m - 2$, the number of 1-transitions is bounded by $2m - 4$.

Point 3. The number of forward transitions is $m + 1$, thus the whole number of transitions is bounded by $4m - 5$. \square

Proof (Of Lemma 2). Searching for p in t using the forward automaton of p can be easily done reading all symbols of the text one after the other. But at each state one must identify the right outgoing transition, which normally requires to search in a list or an AVL tree. This would add a polylog factor to all integer reading and thus the complexity would be of the form $O(n \cdot \text{polylog}(m))$.

However, the structure of the forward automaton combined with the fact that we imposed all outgoing transitions of each node to be sorted increasingly to the length of the transition allow us to amortize the search complexity of the searching phase along the permutation. The resulting search phase complexity is $O(n)$ time. Indeed, let us search t through the automaton, reading one symbol at a time reaching a current state x . Let us assume we read the text until position i and we want to match t_{i+1} . We test if t_{i+1} belongs to the interval $[i, j]$ labeling $x + 1 = \delta(x, [i, j])$ if $x < m$. If yes, we follow this forward transition. If not, we test each backward transition from x in increasing length order.

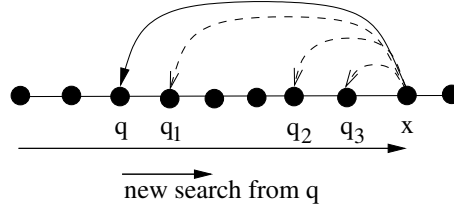


Fig. 3. Amortized complexity of the forward search. The search starts again from q . On this instance $l = 3$ and $q + 3 < x$.

The important point to notice is that after having identified the right backward transition from x for t_{i+1} reaching state q (there must be one), the search for t_{i+2} starts from $q < x$. Moreover, we associate all l transitions $q_k = \delta_k(x, [i, j])$ touched before finding the right one to its ending state which verifies $q < q_k < x$. Thus $q + l < x$. This point is illustrated in Figure 3. As the search starts again from q and that at most one forward transition is passed through by text symbol, the total number of forward and backward transitions touched or passed through when reading the whole text $t = t_1 \dots t_n$ is thus bounded by $2n$. \square

Proof (Proof of lemma 3). Exactly as in the case of a classical text, we amortize the complexity of the search over the number of transitions we pass through and the number of reinitialisations of the search we do if no more failure transition is available. Each time we pass through a failure transition, we decrease the state from where we will go on the search if the state is validated. Thus, there can be at most as many failure transitions passed through during the whole reading of the text as the number of forward transitions that has been passed through. Since this number is at most the size of the text, the total number of transitions touched is at most $2n$. Then, if after a descent from failure transition to failure transition no more outgoing transition exists, we reinitialise the search to state

1. Thus there are at most n such reinitialisations and the total complexity of transitions and states touched is bounded by $3n$. \square

Proof (Of Lemma 4). Before processing, the pattern we first reduce the range of the keys from $[n]$ to $[m]$. This is done in deterministic $O(m \log \log m)$ times by first sorting the keys using the fastest integer sorting algorithm due to Han [8], and then replacing each key by its rank obtained from the sorting.

We then process the pattern in left-to-right in m steps and at each step j determine the failure and forward transitions outgoing of state j . We use two predecessor data structures that require $O(m)$ words of space and support insert, delete and query operations (a query operation returns both the predecessor and the successor) in (worst-case) time $O(\log \log m)$. As we move forward in the pattern, we insert each symbol in both predecessor data structures (except for the first symbol which is only inserted in the first predecessor data structure). The difference between the two predecessor data structures is that the first one will only get insertions while the second one can also get deletions. The first is used to determine forward transitions while the second one is used to determine failure transitions.

We now show how we determine the transitions at each step j . The forward transitions connecting state j to state $j + 1$ is labeled by $\text{rep}(p, j + 1)$. The latter is determined by doing a predecessor search for p_{j+1} on the first predecessor data structure. This gives us both the predecessor and successor of p_{j+1} among $p_1 \dots p_j$ which is exactly $\text{rep}(p, j + 1)$.

The failure transition is determined in the following way. If the target state of the failure transitions of state $j - 1$ is state i . Then we do a predecessor query on the the second predecessor data structure. If the pair of returned prefixes is precisely $\text{rep}(p, i + 1)$, then we can make $i + 1$ as a target for state j . Otherwise we take the failure transition of state $j - 1$. If that transitions leads to a state k , then we remove the symbols $p_{j-i} \dots p_{j-k}$ from the second predecessor data structure. \square

Proof (Of Property 3). We first build the Morris-Pratt representation in $O(m \log \log m)$ time. We then consider each state $x > 0$ corresponding to the $p_1 \dots p_x$ from left to right and for each such state we expand its backward transitions. Let us sort all $p_i, 1 \leq i \leq x$ and consider the resulting order $p_{i_0} = -\infty < p_{i_1} < \dots < p_{i_k} < +\infty = p_{i_{k+1}}$. We build one outgoing transition for each interval $[p_{i_j}, p_{i_{j+1}}]$,

excepted if $p_{i_{j+1}} = p_{i_j} + 1$. This transition is computed as follows. Let q be the image state of the failure transition from x . We pick a value z in $[p_{i_j}, p_{i_{j+1}}]$ and search for z from q . Let q' be the new state reached. We create a backward transition from x to q' labeled $[p_{i_j}, p_{i_{j+1}}]$. After this process we created at most m^2 edges in at most $O(m^2 \log \log m)$ time.

We now merge backward transitions from the same state to the same state that are labeled by consecutive intervals. This required at most $O(m^2)$ time. The whole algorithm thus requires $O(m^2 \log \log m)$ time. \square

Proof (Of Theorem 3). We now give the details of the construction of the failure transitions in the \mathcal{AC} automaton. The construction of the forward transitions has already been explained in section 4.

In order to build the failure transitions we decompose the trie into r layers. The first layer consists in the nodes of the trie that represent prefixes of length 1. The second layer consist in all the nodes that represent prefixes of length 2, etc.

Next, we will reuse the same algorithm that was used in 4 to build the \mathcal{MP} automaton but adapted to work on the \mathcal{AC} automaton. However, instead of using a single predecessor data structure we will use multiple predecessor data structures and attach a pointer to a predecessor data structure at each trie node.

A node of the original non compacted trie will share the same predecessor with its parent, iff it is the only child of its parent. The following building phases will no longer reuse the normalized patterns, but instead reuse the original patterns. To each node, we attach a pointer to one of the original patterns. More precisely if a node has a single child, then his pattern pointer will be the same as its (only) child pattern pointer. If a node has more than one child (in which case it is called a *branching node*), then it will point to the shortest pattern in its subtree. If a node is a leaf then it will directly point to the corresponding pattern. A predecessor data structure of a node whose pattern pointer points to a pattern of length u will have capacity to hold u keys from universe u and thus will use $O(u)$ space. This is justified by the fact that the predecessor data structure will only hold at most u elements of the patterns and each element value is at most u (recall that the pattern is a permutation of length u).

In order to bound the total number of predecessor data structures and their total size, we consider a compacted version of the trie (Patricia trie), where each node with a single child is merged with that single child. A node in the original (non-compacted) trie with two or more children is called *branching node*. It is clear that the set of nodes of a patricia (compacted) trie are precisely the branching nodes and the leaves of the original trie.

It is a well known fact that a Patricia trie with r leaves has at most $2r - 1$ nodes in total. Thus the total number of predecessor data structures will be upper

bounded by $2r - 1$. During the building if a node at layer t has a single child, then that single child at level $t + 1$ will inherit the predecessor data structure of its parent. Otherwise if the node v at level t has two or more children at level $t + 1$, then a predecessor data structure is created for each child u . Then if the predecessor data structure of v contains exactly k elements, those elements are precisely $x_{t+1-k} \dots x_t$, where x is string pointed by v . We will insert the k elements $y_{t+1-k} \dots y_t$ into the predecessor data structure of u , where y the string pointed by u .

In order to bound the total space used by the predecessor data structures, we notice that the total capacities of all predecessor data structures is $O(m)$. This can easily be proved. Because we know that the total length of all patterns is bounded by m , we will also know that the total cumulative length of all strings pointed by branching node is also upper bounded by m . This is because precisely the pointed strings are precisely the shortest strings in the subtrees rooted by the branching node. The same holds for the leaves as the capacities of their respective predecessor data structures will be no more than the total length of the patterns that correspond to the leaves which is $O(m)$.

We finally need to bound the total construction time which is dominated by the operations on the predecessor data structures. The time is clearly bounded by $O(m \log \log r)$. This is by a straightforward argument: as the total sum of the pointed strings is $O(m)$, and we know that each element of a pointed string can only be inserted or deleted once, and furthermore each insert/delete cost precisely $O(\log \log r)$ worst-case time, we conclude that the total time spent in the predecessor data structure is worst case $O(m \log \log r)$. \square

Proof (Of Lemma 5). Let A be an algorithm to search for p in t running in $O(l)$ time. It can be converted in an algorithm to search for p inside all $s_i(t)$ also running in $O(t)$ since: (a) it suffices to remove all occurrences overlapping two segments and occurrences in the last few remaining symbols of t out of a segment; and (b) in $O(l)$ time, only at most $O(l)$ such occurrences can be reported, so only $O(l)$ occurrences might have to be discarded; and (c) testing if an occurrence is overlapping two segments can be done in constant $O(l)$ time. The extra work required to remove all overlapping occurrences is therefore also $O(l)$, and thus A can be converted in an $O(l)$ algorithm to search for p inside all segments $s_i(t)$. This implies that a lower bound for this last problem is also a lower bound for A .

Proof (Of Lemma 6). All segments $s_i(t)$ are identically distributed, independently of each other. Thus the average time for searching for p in any segment is the same. As the expected time is the sum of the expected time to search for p in all segments, the sum commutes and the expected time becomes $\lfloor n/(2m-1) \rfloor$ times the average expected time to search for p in any segment. \square

Proof (Of Lemma 7). Let $1 \leq i_1 < i_2 \dots < i_\ell \leq m$ be the position of the accesses. For $0 \leq j \leq d$, we define

$$B_j = \{b \mid b \in \{1, 2, \dots, m\} \text{ and } j + b = i_t \text{ for some } 1 \leq t \leq \ell\}.$$

Note that $|B_j| \leq \ell$ for $1 \leq j \leq d$. Also, for any $p \in \mathcal{P}_m(\ell)$, since it is canceled by the ℓ accesses considering isomorphic orders, for all shift j there is a mismatch, *i.e.* there exists two positions $k, \ell \in B_j$ such that $p[k] > p[\ell]$ and $t[j+k] < t[j+\ell]$. We then show that we can find $J \subset \{0, 1, \dots, d\}$, $|J| = \lceil d/\ell^2 \rceil$, such that $B_{j_1} \cap B_{j_2} = \emptyset$ for $j_1 \neq j_2$ in J .

We use a greedy procedure to find J . Let $j_1 = 0$. Inductively, suppose that we have found $j_1 \dots j_{k-1}$. Then j_k is obtained by finding the smallest j such that B_j is disjoint from the unions of the previous positions we have already chosen, namely $B = B_{j_1} \cup B_{j_2} \cup \dots B_{j_{k-1}}$. We claim that this procedure allows us to find at least $\lceil d/\ell^2 \rceil$ such sets. We prove in fact that $j_k \leq \ell^2(k-1)$ as long as $\ell^2(k-1) \leq d$. Observe that B contains at most $\ell(s-1)$ positions. We thus claim that at least one of the sets in $\mathcal{F} = \{B_0, B_1, \dots, B_{\ell^2(s-1)}\}$ is disjoint from B . If not, for each r , $0 \leq r \leq \ell^2(s-1)$ there exists a pair (b, i_t) such that $b \in B_r \cap B$ and $r+b = i_t$ for some $1 \leq t \leq \ell$. So there must exist at least $\ell^2(s-1)+1$ such pairs, one for each set B_r . But the total number of such pair is no more than $|B| \cdot \ell \leq \ell^2(s-1)$, a contradiction.

Now take $J \subset \{0, 1, \dots, d\}$, $|J| = \lceil d/\ell^2 \rceil$, such that $B_{j_1} \cap B_{j_2} = \emptyset$ for $j_1 \neq j_2$ in J . To prove the lemma, consider a random pattern p from \mathcal{S}_m (the set of permutations of size m). Then for all shift $j \in \{0 \dots d\}$, there is a mismatch. So

$$\begin{aligned} P(p \in \mathcal{P}_m(\psi)) &= P(\forall j \in \{1 \dots d\}, \text{ there is a mismatch}) \\ &\leq P(\forall j \in J, \text{ there is a mismatch}). \end{aligned}$$

Notice that for each $j \in \{1 \dots d\}$, the probability that there is no mismatch with p at shift j is $\frac{1}{|B_j|!}$ which is the probability that the permutation formed by the non- \star symbol is the good one. Since all the sets B_j for $j \in J$ are disjoint, we have

$$\begin{aligned} P(p \in \mathcal{P}_m(\ell)) &\leq \prod_{j \in J} P(\text{there is a mismatch at shift } j) \\ &\leq \prod_{j \in J} \left(1 - \frac{1}{|B_j|!}\right) \leq \left(1 - \frac{1}{\ell!}\right)^{\lceil \frac{d}{\ell^2} \rceil} \end{aligned}$$

concluding the proof since $|\mathcal{S}_m| = m!$. \square

Here we prove that $\ell = \frac{b \log m}{\log \log m}$ with some $b = 1 + o(1)$ verify the following inequalities for m large enough: $\left\lceil \frac{m-1}{\ell^2} \right\rceil \times \frac{1}{\ell!} \leq 1/10$ (ineq.1) and

$$\frac{195}{200} \leq 1 - \left\lceil \frac{m-1}{\ell^2} \right\rceil \times \frac{1}{\ell!} \leq \frac{99}{100}.$$

Let's recall that the Gamma function of Euler Γ is an increasing bijection from $\mathbb{R}_{\geq 2}$ to $\mathbb{R}_{\geq 1}$ verifying that $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$.

Thus the function $s \mapsto s^2 \Gamma(s+1)$ is an increasing bijection from $\mathbb{R}_{\geq 1}$ to $\mathbb{R}_{\geq 1}$.

For all $m \geq 1$, this allows to define $s \in \mathbb{R}_{\geq 1}$ such that $s^2 \Gamma(s+1) = 50m$.

Thus $s \rightarrow \infty$ when $m \rightarrow \infty$.

Then we set $b = s \times \frac{\log \log m}{\log m}$.

Taking $\ell = s$, then we have $\ell = \frac{b \log m}{\log \log m}$.

Let us prove that ℓ satisfied the desired inequalities.

We have $\ell^2 \times \ell! = s^2 \Gamma(s+1) = 50m$.

Thus $\left\lceil \frac{m-1}{\ell^2} \right\rceil \times \frac{1}{\ell!} \leq \frac{m}{50m} \leq \frac{1}{50}$ and $\left\lceil \frac{m-1}{\ell^2} \right\rceil \times \frac{1}{\ell!} \geq \frac{m-1}{50m} \geq \frac{1}{100}$ for m large enough. This proves the desired inequalities since $1/50 \leq 1/40 = 1 - 195/200 \leq 1/10$.

Let us prove now that $b = 1 + o(1)$.

By the Stirling inequality, we have that $F(s) < \Gamma(s+1) < 2F(s)$ with $F(s) = \left(\frac{s}{e}\right)^s \sqrt{2\pi s}$.

Thus $\sqrt{2\pi} \exp(G(s)) < s^2 \Gamma(s+1) < 2\sqrt{2\pi} \exp(G(s))$

with $G(s) = -s + \left(s + \frac{5}{2}\right) \log(s)$.

We deduce that $\frac{25m}{\sqrt{2\pi}} < \exp(G(s)) < \frac{50m}{\sqrt{2\pi}}$.

Therefore $\log\left(\frac{25}{\sqrt{2\pi}}\right) + \log m < G(s) < \log\left(\frac{50}{\sqrt{2\pi}}\right) + \log m$.

Recall that $f(m) \sim g(m)$ means that $f(m) = g(m) + o(g(m))$ when $m \rightarrow \infty$.

Thus we have $G(s) \sim \log m$.

It is then enough to prove that $G(s) \sim b \log m$. Indeed this imply $b \sim 1$, i.e., $b = 1 + o(1)$.

But $G(s) = -s + \left(s + \frac{5}{2}\right) \log(s) = s \log s + o(s \log s)$.

Thus $s = o(G(s))$, i.e., $s = o(\log m)$.

Since $s = \frac{b \log m}{\log \log m}$, this means that $b = o(\log \log m)$

Moreover $\log s = \log b + \log \log m - \log \log \log m = \log \log m + o(\log \log m)$.

As $G(s) = -s + \left(s + \frac{5}{2}\right) \log(s)$ we then have:

$$G(s) = -\frac{b \log m}{\log \log m} + \left(\frac{b \log m}{\log \log m} + \frac{5}{2}\right) \left(\log \log m + o(\log \log m)\right).$$

Thus $G(s) \sim b \log m$, concluding the proof.